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TABLES OF POLYNOMIALS FOR GRADUATION BY MITSCHERLICH'S EQUATION

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1. Introduction

Pimental Gomes and Malavolta (1949) indicated a procedure for graduating experimental data by Mitscherlich's equation according to the method of least squares, which is equivalent, in the case of interest here, to the method of maximum likelihood. Unhappily, however, the procedure was quite laborious.

Now, however, we are presenting a series of six tables of functions which permit rapid and precise graduation.

2. The Polynomials Tabulated

Let us consider the case of an experiment with replication and four treatments with the levels q, 2q, 3q, and 4q of fertiliser or manure. We prefer to take this case as a basic example because the number of treatments is not excessive and leads to an analysis of variance with two degrees of freedom for regression as represented by Mitscherlich's law (Pimental Gomes (1950a) and (1950b)).

The equation to be solved is, then

$$\begin{vmatrix} \Sigma y_i & n & \Sigma 10^{-c\,x_i} \\ \Sigma x_i y_i 10^{-c\,x_i} & \Sigma x_i 10^{-c\,x_i} & \Sigma x_i 10^{-2c\,x_i} \\ \Sigma y_i\, 10^{-c\,x_i} & \Sigma 10^{-c\,x_i} & \Sigma 10^{-2c\,x_i} \end{vmatrix} = 0$$

where $x_1 = 0$, $x_2 = q$, $x_3 = 2q$, $x_4 = 3q$, $x_5 = 4q$ and y_1, y_2, y_3, y_4 and y_5 represent the yields obtained with the corresponding levels of x. We can take $s = 10^{-cq}$ and obtain the new equation

$$\begin{vmatrix} \Sigma y_i & n & \Sigma s^{i-1} \\ \Sigma x_i y_i s^{i-1} & \Sigma x_i s^{i-1} & \Sigma x_i s^{2i-2} \\ \Sigma y_i s^{i-1} & \Sigma s^{i-1} & \Sigma s^{2i-2} \end{vmatrix} = 0$$

We can divide the second row by s and fix $n = 5$. Also, for simplicity, we take q as unity, so that $x_1=0$, $x_2=1$, $x_3=2$, $x_4=3$, $x_5=4$. The determinant, written out in full, will be the following:

$$\begin{vmatrix} (y_1+y_2+y_3+y_4+y_5) & 5 & (1+s+s^2+s^3+s^4) \\ (y_2+2sy_3+3s^2y_4+4s^3y_5) & (1+2s+3s^2+4s^3) & (s+2s^3+3s^5+4s^7) \\ (y_1+sy_2+s^2y_3+s^3y_4+s^4y_5) & (1+s+s^2+s^3+s^4) & (1+s^2+s^4+s^6+s^8) \end{vmatrix} = 0$$

The development of this determinant gives us:

$$y_1(3s - 3s^2 + 3s^3 - 9s^4 + 6s^5 - 8s^6 + 13s^7 - 3s^8 - s^9 - s^{10}) +$$
$$y_2(-3 + 2s + 3s^2 + s^3 + s^4 - 6s^5 + 3s^6 - 8s^7 + 9s^8 - s^9 - s^{10}) +$$
$$y_3(1 - 7s + 6s^2 + s^3 + 3s^4 - 3s^6 - s^7 - 6s^8 + 7s^9 - s^{10}) +$$
$$y_4(1 + s - 9s^2 + 8s^3 - 3s^4 + 6s^5 - s^6 - s^7 - 3s^8 - 2s^9 + 3s^{10}) +$$
$$y_5(1 + s + 3s^2 - 13s^3 + 8s^4 - 6s^5 + 9s^6 - 3s^7 + 3s^8 - 3s^9) = 0$$

These five polynomials, are, however, divisible by $(s-1)^3$, as shown by Nogueira (1950). Performing the division we obtain:

$$y_1(3s + 6s^2 + 12s^3 + 12s^4 + 12s^5 + 4s^6 + s^7) +$$

$$y_2(-3 - 7s - 9s^2 - 8s^3 - 3s^4 + 4s^6 + s^7) +$$

$$y_3(1 - 4s - 9s^2 - 13s^3 - 13s^4 - 9s^5 - 4s^6 + s^7) +$$

$$y_4(1 + 4s - 3s^3 - 8s^4 - 9s^5 - 7s^6 - 3s^7) +$$

$$y_5(1 + 4s + 12s^2 + 12s^3 + 12s^4 + 6s^5 + 3s^6) = 0$$

These polynomials, the coefficients of the various values of y, are the ones we must tabulate in order to facilitate the solution of the equation.

The values of s, which are of interest for the tabulations, are those running from 0 to 1, for, since

$$s = 10^{-c},$$

once we take q as unity and $c > 0$, it is clear that the following holds

$$0 < s < 1.$$

The calculation of the tables could not have been accomplished without the inestimable assistance of the personnel and specialized equipment of the Statistical Section of "Luis de Queiros", kindly furnished through Prof. F. G. Brieger.

3. An Illustrative Example

In an experiment in liming wheat conducted in Ponta Grossa, Parana, by the Ministry of Agriculture, slaked lime (calcium hydroxide) was applied at levels of 0, 2, 4, 6 and 8 metric tons per hectare. A 5 x 5 Latin square was used. The lime was applied in 1940 and the wheat was grown from 1940 to 1948 in the same plots. The data for 1942 are as shown below.

Level of lime	0	2	4	6	8
Mean yield (kg/ha)	984	1386	1458	1486	1464

The equation to be solved will be, then,

$$R(z) = 984\ J_1\ (z) + 1386\ J_2\ (z) + 1458\ J_3\ (z) + 1486\ J_4\ (z) + 1464\ J_5\ (z) = 0$$

where $J_1\ (z)$, $J_2\ (z)$, etc. are the tabulated polynomials.

For $z = 0$ the tables give

$$J_1\ (z) = 0,\ J_2\ (z) = -3,\ J_3\ (z) = J_4\ (z) = J_5\ (z) = 1$$

Accordingly,

$$R\ (0) = 984 \times 0 - 1386 \times 3 + 1458 \times 1 + 1486 \times 1 + 1464 \times 1 = 250$$

For $z = 1$ we obtain in a similar manner

$$R(1) = 984 \times 50 - 1386 \times 25 - 1458 \times 50 - 1486 \times 25 + 1465 \times 50 = -22300$$ *

Since $R(0)$ and $R(1)$ have opposite signs, the root sought necessarily lies between 0 and 1. It is easy to see that the root should be closer to 0 than to 1. Let us, then, take, for example, $z = .4$. We find, by means of the tables,

$$R(0.4) = 984 \times 3.376 - 1386 \times 7.811 - 1458 \times 3.312 + 1486 \times 2.077 + 1464 \times 5.669 = -947.120$$

Accordingly, the root is between zero and .4, being closer to zero. Let us take, then, $z = .1$. We obtain, again from the tables,

$$R\ (.1) = 984 \times 0.373 - 1386 \times 3.798 + 1458 \times 0.496 + 1486 \times 1.396 + 1464 \times 1.533 = 144.940.$$

Since $R(0.1)$ is positive and $R(0.4)$ is negative, the root will be between 0.1 and 0.4. Therefore, let $z = 0.25$. Then

$$R(0.25) = 984 \times 1.372 - 1386 \times 5.448 - 1458 \times 0.826 + 1486 \times 1.911 + 1464 \times 2.991 = -185.618.$$

So the root will be between 0.1 and 0.25.

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*1465 should read 1464

Up to this point it would have been possible to make the calculations with only two decimals or even with only one.

Now that the root has been located within a rather small interval, we can try to determine it by more precise methods.

When z varies from 0.1 to 0.25; that is, when it takes an increment of 0.15, the increment in R (z) is

144.940 - (-186.618) = 331.558 ~ 332.

We set up the proportion

$$\frac{0.15}{332} = \frac{x}{145}$$

and obtain x = 0.066. So we take, as a better estimate of the root, z = 0.1 + 0.066 = 0.166. The true root will be in the vicinity of this value. Let us take z = 0.16. We obtain

$$R(0.16) = 45.130.$$

The root will be, then, between 0.16 and 0.25. A new proportion

$$\frac{0.09}{x} = \frac{232}{45}$$

gives us x = 0.02, so z = 0.16 + 0.02 = 0.18. We find, however,

$$R(0.18) = 3.600.$$

Therefore, the root will be between 0.18 and 0.25, very close to the first value; the new porportion indicates that the root will be between 0.18 and 0.19. Let us take z = 0.19; we will obtain

$$R(0.19) = -18.032$$

The new propertion will be

$$\frac{0.01}{x} = \frac{21.63}{3.6}$$

so $x = 0.0017$. Thus, the root will be approximately

$$z = 0.18 + 0.0017 = 0.1817.$$

We take

$$10^{-2c} = 0.1817$$

and obtain

$$c = \frac{\operatorname{colog} 0.1817}{2} = 0.3703$$

4. The Computation of A

The value of A is given by the formula

$$A = \frac{\begin{matrix} \Sigma y & \Sigma z^{i-1} \\ \Sigma y z^{i-1} & \Sigma z^{2i-2} \end{matrix}}{\begin{matrix} 5 & \Sigma z^{i-1} \\ \Sigma z^{i-1} & \Sigma z^{2i-2} \end{matrix}}$$

or

$$(4.1)\quad A = \frac{1}{F(z)} \begin{matrix} y_1(-z-z^3+z^6+z^8)+y_2(1-z-z^3-z^5+z^6+z^8) \\ +\, y_3(1-z^3-z^5+z^8)+y_4(1+z^2-z^3-z^5-z^7+z^8)+ \\ y_5(1+z^2-z^5-z^7) \end{matrix}$$

where

$$F(z) = 4 - 2z + 2z^2 - 4z^3 - 4z^5 + 2z^6 - 2z^7 + 4z^8$$

This last polynomial was also tabulated. The other polynomials which appear in (4.1) are easily calculated, particularly if Barlow's tables are used.

In the case of the foregoing example we have

$$F(0.18) = 3.6808$$

$$F(0.19) = 3.6639$$

By interpolation we find

$$F(0.1817) = 3.6779$$

so we obtain

$$A = \frac{1}{3.6779} \begin{array}{l} - 984 \times 0.18766 + 1386 \times 0.81214 \\ + 1458 \times 0.99380 + 1486 \times 1.02681 \\ + 1464 \times 1.03281 \end{array} = 1475.8$$

Finally,

$$b = \frac{1}{c} \log \frac{A(1 + z + z^2 + z^3 + z^4)}{5A - (y_1 + y_2 + y_3 + y_4 + y_5)}$$

$$= \frac{1}{c} \log 3.0022$$

$$= 1.2893.$$

Thus, Mitscherlich's equation for the case being studied is

$$y = 1475.8 \,[1 - 10^{-0.3703(x + 1.2893)}]$$

5. An Important Property

The polynomials tabled

$$J_1(z) = 3z + 6z^2 + 12z^3 + 12z^4 + 12z^5 + 4z^6 + z^7,$$

$$J_2(z) = -3 - 7z - 9z^2 - 8z^3 - 3z^4 + 4z^6 + z^7,$$

$$J_3(z) = 1 - 4z - 9z^2 - 13z^3 - 13z^4 - 9z^5 - 4z^6 + z^7,$$

$$J_4(z) = 1 + 4z - 3z^3 - 8z^4 - 9z^5 - 7z^6 - 3z^7,$$

$$J_5(z) = 1 + 4z + 12z^2 + 12z^3 + 12z^4 + 6z^5 + 3z^6$$

have the important property that

$$(5.1) \quad J_1(z) + J_2(z) + J_3(z) + J_4(z) + J_5(z) = 0.$$

This property allowed us to verify the computations in the tables of these polynomials. However, due to the approximations involved in showing the tables to only three decimals, there is, in some cases, a small difference of .001 in the check by the identity (5.1).

6. Tables of the Polynomials

The polynomials tabulated are:

$$J_1(z) = 3z + 6z^2 + 12z^3 + 12z^4 + 12z^5 + 4z^6 + z^7,$$

$$J_2(z) = -3 - 7z - 9z^2 - 8z^3 - 3z^4 + 4z^6 + z^7,$$

$$J_3(z) = 1 - 4z - 9z^2 - 13z^3 - 13z^4 - 9z^5 - 4z^6 + z^7,$$

$$J_4(z) = 1 \;[illegible]\; 4z - 5z^3 - 8z^4 - 9z^5 - 7z^6 - 3z^7,$$

$$J_5(z) = 1 + 4z + 12z^2 + 12z^3 + 12z^4 + 6z^5 + 3z^6$$

$$P(z) = 4 - 2z + 2z^2 - 4z^3 - 4z^5 + 2z^6 - 2z^7 + 4z^8.$$

z	J1(z)	J2(z)	J3(z)	J4(z)	J5(z)	P(z)
0.00	0.000	-3.000	1.000	1,000	1.000	4.0000
0.01	0,031	-3,071	0.959	1,040	1 041	3.9802
0.02	0,062	-3.144	0.916	1,080	1.085	3.9608
0,03	0,096	-3,218	0,872	1,120	1,131	3,9417
0.04	0.130	-3,295	0 825	1.160	1,180	3,9229
0.05	0.166	-3,374	0.776	1.200	1,232	3.9045
0.06	0.204	-3.454	0.725	1,239	1,286	3,8863
0,07	0.244	-3.537	0.671	1.279	1.343	3,8684
0,08	0,285	-3.622	0.615	1,318	1.403	3,8507
0.09	0.328	-3,709	0 557	1,357	1.467	3,8333
0.10	0.373	-3,798	[illegible]	1,396	1.533	3,8160
0,11	0.421	-3,890	0.432	1,435	1.603	3.7968
0,12	0.470	-3.984	0.365	1.473	1,676	3.7818
0.13	0,522	-4,081	0,295	1.511	1,753	3,7649
0,14	0.576	-4.179	0.222	1.548	1.833	3,7480
0,15	0.633	-4.281	0,146	1.585	1.917	3.7312
0.16	0,692	-4.385	0.067	1,621	2.005	3.7144
0.17	0,754	-4,492	-0.016	1,657	2.097	3,6976
0.18	0.819	-4.601	-0.103	1,692	2,193	3.6808
0,19	0.888	-4,713	-0.193	1,726	2,293	3,6639
0,20	0,959	-4,829	-0.288	1,760	2,397	3 6468
0,21	1 034	-4,946	-0.387	1,792	2,506	3,6297
0 22	1.113	-5,067	-0.490	1,824	2,620	3,6123
0,23	1,195	-5,191	-0,597	1,854	2,739	3,5948

z	J1(z)	J2(z)	J3(z)	J4(z)	J5(z)	P(z)
0.24	1 282	-5.318	-0.709	1,883	2,862	3.5771
0,25	1.372	-5 448	-0,826	1,911	2.991	3.5590
0,26	1,467	-5.581	-0,948	1,938	3,125	3.5407
0,27	1,566	-5,718	-1.075	1 962	3,265	3,5220
0,28	1,670	-5,858	-1 208	1,986	3,410	3.5030
0,29	1,779	-6.001	-1,347	2.007	3.561	3.4835
0,30	1.893	-6.147	-1,491	2,027	3,718	3 4636
0,31	2.013	-6.297	-1,641	2,044	3.881	3,4432
0.32	2.138	-6,451	-1.793	2.059	4 051	3.4222
0.33	2.270	-6.608	-1,951	2,072	4.228	3,4007
0.34	2,407	-6,768	-2.132	2,082	4.411	3.3786
0,35	2.551	-6 933	-2,309	2,089	4,602	3,3559
0.36	2.701	-7.101	-2,494	2,094	4.800	3.3323
0.37	2.859	-7.272	-2,686	2,095	5.003	3,3081
0.38	3,023	-7.448	-2 886	2.093	5,218	3.2831
0.39	3,196	-7 627	-3.095	2,087	5,439	3.2573
0.40	3.376	-7.811	-3 312	2,077	5,669	3 2306
0.41	3.565	-7.998	-3,538	2,064	5.907	3.2030
0.42	3.762	-8,189	-3.773	2,046	6.154	3.1744
0,43	3.968	-8.385	-4,017	2,023	6,410	3,1448
0.44	4,184	-8 584	-4.271	1,996	6,676	3.1142
0.45	4,409	-8.788	-4.536	1,963	6,951	3 0825
0.46	4,644	-8.995	-4,811	1,925	7,237	3.0497
0,47	4.890	-9.207	5,097	1.881	7.532	3,0157
0.48	5.147	-9.423	-5.394	1.831	7.839	2.9805
0.49	5,415	-9.643	-5,703	1,774	8,156	2.9449
0.50	5,695	-9,867	-6,023	1.711	8,484	2.9062
0,51	5.988	-10.096	-6 357	1,640	8,825	2,8671
0.52	6.293	-10.328	-6.703	1,562	9,177	2.8266
0.53	6,611	-10.565	-7,063	1,475	9.542	2 7848
0.54	6.943	-10.807	-7.436	1.380	9.919	2.7414
0.55	7,289	-11.052	-7.823	1,276	10,310	2.6966
0.56	7 651	-11.302	-8.226	1,163	10,714	2.6503
0 57	8,027	-11 556	-8.643	1,040	11,132	2,6024
0.58	8.420	-11 814	-9,076	0.906	11,564	2 5530
0.59	8,829	-12,076	-9.525	0,761	12 011	2.5020
0.60	9,255	-12 342	-9 991	0.605	12,474	2.4495
0.61	9.699	-12.613	-10 474	0.436	12,952	2 3953
0.62	10.161	-12.867	-10,975	0,255	13,446	2.3395
0.63	10.643	-13,166	-11,494	0,061	13,957	2.2822
0 64	11.144	-13,418	-12.033	-0.148	14.485	2.2232
0,65	11,666	-13.734	-12.590	-0,371	15,030	2.1626
0 66	12,208	-14,024	-13.168	-0.610	15,593	2,1005
0.67	12,773	-14.318	-13.756	-0,865	16,176	2.0368
0.68	13.361	-14.616	-14,386	-1,136	16,777	1.9717
0.69	13,972	-14.917	-15.027	-1.425	17.398	[illegible]
0.70	14,607	[illegible]	-15,691	-1,733	18,039	[illegible]

z	J1(z)	J2(z)	J3(z)	J4(z)	J5(z)	P(z)
0,71	15,267	-15,529	-16,379	-2,060	18,700	1,7675
0,72	15,954	-15,840	-17,090	-2,407	19,384	1,6967
0,73	16,667	-16,154	-17,826	-2,775	20,089	1,6248
0,74	17,408	-16,471	-18,587	-3,166	20,816	1,5518
0,75	18,177	-16,791	-19,374	-3,579	21,567	1,4777
0,76	18,977	-17,114	-20,138	-4,016	22,342	1,4028
0,77	19,806	-17,439	-21,030	-4,478	23,141	1,327[illegible]
0,78	20,668	-17,766	-21,900	-4,967	23,965	1,2509
0,79	21,562	-18,095	22,800	-5,482	24,815	1,1742
0,80	22,490	-18,427	-23,729	-6,026	25,692	1,0972
0,81	23,452	-18,759	-24,689	-6,590	26,595	1,0203
0,82	24,450	-19,094	-25,680	-7,204	27,527	0,9435
0,83	25,486	-19,429	-26,704	-7,840	28,487	0,867[illegible]
0,84	26,559	-19,765	-27,762	-8,509	29,477	0,7915
0,85	27,672	-20,102	-28,854	-9,214	30,497	0,7168
0,86	28,826	-20,440	-29,980	-9,954	31,548	0,6434
0,87	30,021	-20,777	-31,143	-10,732	32,631	0,5717
0,88	31,259	-21,114	-32,343	-11,548	33,746	0,5020
0,89	32,542	-21,451	-33,581	-12,406	34,895	0,1347
0,90	33,871	-21,786	-34,858	-13,305	36,078	0,3702
0,91	35,247	-22,120	-36,175	-14,248	37,297	0,3090
0,92	36,672	-22,453	-37,533	-15,237	38,551	0,2516
0,93	38,146	-22,783	-38,933	-16,273	39,843	0,1985
0,94	39,672	-23,111	-40,376	-17,357	41,172	0,1503
0,95	41,252	-23,436	-41,863	-18,4[illegible]3	42,541	0,1076
0,96	42,886	-23,758	-43,305	-19,681	43,949	0,0709
0,97	44,576	-24,075	-44,974	-20,924	45,398	0,0411
0,98	46,324	-24,389	-46,600	-22,223	46,889	0,0188
0,99	48,131	-24,697	-48,275	-23,581	48,422	0,0049
1,00	50,000	-25,000	-50,000	-25,000	50,000	0,0000

7. An Important Observation

Equation (2.1), after arrangement with respect to z, gives:

$$(y_1 + y_2 + y_3 - y_4)\, z^7 + (4y_1 + 4y_2 - 4y_3 - 7y_4 + 3y_5)\, z^6 +$$
$$(12y_1 - 9y_3 - 9y_4 + 6y_5)\, z^5 + (12y_1 - 3y_2 - 13y_3 - 8y_4 + 12y_5)\, z^4 +$$
$$(12y_1 - 8y_2 - 13y_3 - 3y_4 + 12y_5)\, z^3 + (6y_1 - 9y_2 - 9y_3 + 12y_5)\, z^2 +$$
$$(3y_1 - 7y_2 - 4y_3 + 4y_4 + 4y_5)\, z + (-3y_2 + y_3 + y_4 + y_5) = 0.$$

This equation must have at least one variation in sign in order for the graduation to be possible. In effect, the absence of such variation indicates the non-existence of a positive root, and therefore the impossibility of finding the root between zero and one, which we require.

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